Black Magic of Calculus

Synth Magic

Preface

This book is kinda a collection of my thoughts, notes and extensive mathematical knowledge while studying Cambridge International Education (CIE) A-level Further Mathematics. Because of the scarcity of resources online. I currently haven't found a concise, clear route of self-studying A-level Further Mathematics on the Internet. So I decided to write this book to help myself and hopefully others.

Moreover, This book is the by-product of my own readings and the extension beyond the syllabus, Maybe I am a bit arrogant,but I don't think the syllabus is trying to teach students math from a ground-up order, so I decided to work it out on my own.

To be clear, I am not a professional mathematician, nor a teacher. I am just a student who wants to share the struggles and thoughts that I had while studying A-level Further Mathematics.

Although I myself is studying CIE A-level Further Mathematics, I believe that this book can also be useful for students studying other examination boards, such as Edexcel and AQA. I belive this book also applies to Singapore GCE A-level students and students who're preparing for their STEP/TMUA/MAT examination.

Also, since I don't wanna this book to be another "dry" textbook, the register and the tone will obviously be informal. and since english is not my first language, you probably will find lots of grammatical and spelling errors. If you do find any, please make an issue or pull request on The github repository of this book°. You can also send me a pull-request if you want to make any contributions.

I would like to thank all the people who're currently reading the book. You guys make my time and effort worth it.

Finally, to Ms.Cassie, thank you for bringing me to the world of mathematics.

Synth Magic

Sep 12, 2025

Thank you all

Here are some of the resources I've read or seek help for, I hereby present my greatest thanks to all of you guys, without you the book is impossible.

The list of resources (ranking in no particular order):

- *Composite integration method* By Zhu Yongyin,Guo Wenxiu (ISBN: 7-5609-2827-7)
- *Problems in mathematical analysis* By B. Demidovich and others (ISBN: 9785030009438)
- *Tide of Indefinite Integrals* By Xu TiaoZi° (Currently not published)

The list of contributors (ranking in no particular order):

- Synth Magic (Founder, main contributor)
- Pieceofmeat (discord)
- RyanCantPvP (discord)
- ALTR (discord)
- If you want to be the one on the list, visit our github repo° and make an issue or pull request

How to use this book

yeah uh pretty much every textbook has these with like beautiful boxes in it, so why don't we add this section as well?

Example Theorem

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$$

Example definition

Natural Number ($\mathbb{N}$) is defined as 1 and its successor

Example lemma

Given two segments a and b, there exists a real number r such that $b = ra$

Here's a tip

When you see a question that asks you to prove something, try to start from the thing that you want to prove.

Here's a note

This is a note box. You can use it to add extra information that is not essential to the main text.

Here's a warning

This is a warning box. You can use it to warn the reader about something.

Here's a caution

This is a caution box. You can use it to caution the reader about something.

Remark

This is a remark box. You can use it to add extra information that is not essential to the main text.

> I didn't say that —— Shing-Tung Yau

Example Problem

Evaluate

$$\int_0^\pi \ln(2+\cos x)\ \mathrm{d}x$$

Example Solution

let $f(a) = \int_0^\pi \ln(1+\cos x)$ where $a \geq 0$, therefore

$$f'(a) = \int_0^\pi \frac{1}{a+\cos x}\ \mathrm{d}x = \frac{\pi}{\sqrt{a^2-1}} \quad (a > 1)$$

Also,

$$f(1) = \int_0^\pi \ln(1+\cos x)\ \mathrm{d}x = \int_0^\pi \ln\left(2\cos^2\frac{x}{2}\right)\ \mathrm{d}x = -\pi\ln 2$$

Hence,

$$f(2) = f(1) + \int_1^2 f'(a)\ \mathrm{d}a = -\pi\ln\left(4-2\sqrt{3}\right)$$

Contents

1. Limits

> I always think that it's a shame on A-level Mathematics that they don't teach limits properly
> —— Justin Yu

Although this is a bit narcissistic, you can't deny that I am spitting fact. Limits aren't taught properly in A-level Mathematics & Further Mathematics.

In fact, limits are barely mentioned in A-level Mathematics & Further Mathematics textbooks, and the notation they use often looks a bit weird and funny. Like come on bro, nobody uses this $x \to 0$ notation anymore.

However, in university and higher level of mathematics, limits is a fundamental concept that is used everywhere.

> **Note**
>
> If you're doing AP,IBDP,JEE and other kinds of standardised test that has a well-defined and detailed definition of limits, you can jump through this chapter. Also this chapter doesn't cover all of the basic limits, it is shrinked to what is useful for A-level examination.

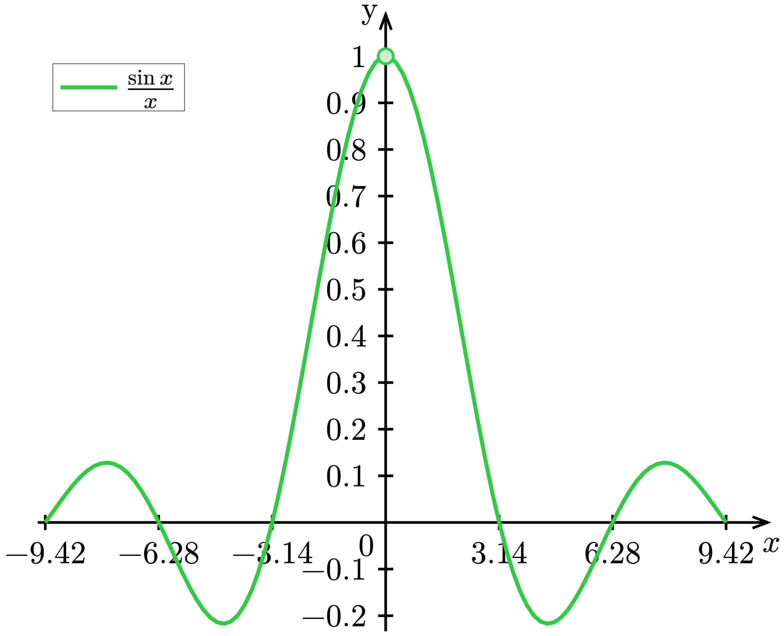
1.1. Why do limits exist anyway?

Limits exists because we wanna study the properties of functions at points that are undefined.

For example, let's take a look at this function:

$$f(x) = \frac{\sin x}{x}$$

Here is the graph of $y = \frac{\sin x}{x}$ and $x = 0$



Obviously, since the denominator is 0 when $x = 0$, the function is undefined at $x = 0$. However, by the graph, we can see that as x approaches 0, $f(x)$ approaches 1. If you still don't get it, here's the table of value of $f(x)$ when x is very close to 0.

x	$\frac{\sin x}{x}$
1	0.841470985
0.1	0.998334166
0.01	0.999983333
0.001	0.999999833

Now It's very clear that as x approaches 0, $f(x)$ approaches 1.

In order to record this kind of tendency of change of function. We use limits. In formal notation, we write this as:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

> **Note**
>
> The whole book will use the formal notation instead of $x \to n$ notation. Even if the extracted original question uses that notation.

Of course, this is just an intuitive explanation of limits. We didn't really define what exactly is a limit, but how can we do this?

1.2. Cauchy's thoughts

Since the foundation of the calculus, mathematicians in many generations have been trying to formally define limits. One of them is Augustin-Louis Cauchy. In his book *Cours d'Analyse* published in 1821, he defined limits as follows:

> When the values successively attributed to the same variable **approach indefinitely** a fixed value, eventually differing from it by **as little as one could wish**, that fixed value is called the limit of all the others. —— Augustin-Louis Cauchy, *Cours d'Analyse*, 1821

Cauchy's definition is pretty good. In fact, it's very close to how we define limits today. However, it still has some problems. First, what does "approach indefinitely" mean? Do we have to approach it continuously or we can approach it in a jumpy manner? Second, what does "as little as one could wish" mean? Does it mean that we can get as close as we want to the limit, or does it mean that we can get infinitely close to the limit? This certianly needs to be clarified.

1.3. Weierstrass's ε-δ definition

It wasn't until 19th centry that Karl Weierstrass gave the modern definition of limits using ε and δ. The definition is as follows:

ε-δ definition of limits Let $f(x)$ be defined on a deleted neighbourhood that contains x_0. If there exists a constant A such that for every $\varepsilon > 0$ (no matter how small), there always exists a positive constant δ such that when x satisfies $0 < |x - x_0| < \delta$, it follows that $|f(x) - A| < \varepsilon$.

Therefore we call A the limit of $f(x)$ as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = A$$

Yeah… this is the offical definition of limits. And to be honest, it looks like gibberish.

1.4. Translation to human readable language

First we need to understand what a "deleted neighbourhood" is, and in order to do that, we need to understand what is a neighbourhood first.

Definition of neighbourhood

Any open interval that it's center is x_0 is called a neighbourhood of x_0.

> **Note**
>
> An open interval is an interval that doesn't include its endpoints. For example, for constant a, b. $a < x < b$ is an open interval and $a \le x \le b$ is a closed interval. For convenience and ~~I am lazy~~, later in this book we will use (a, b) for open intervals and $[a, b]$ for closed intervals.

With this definiton of neighbourhood, we can also define a neighbourhood with a specified radius.

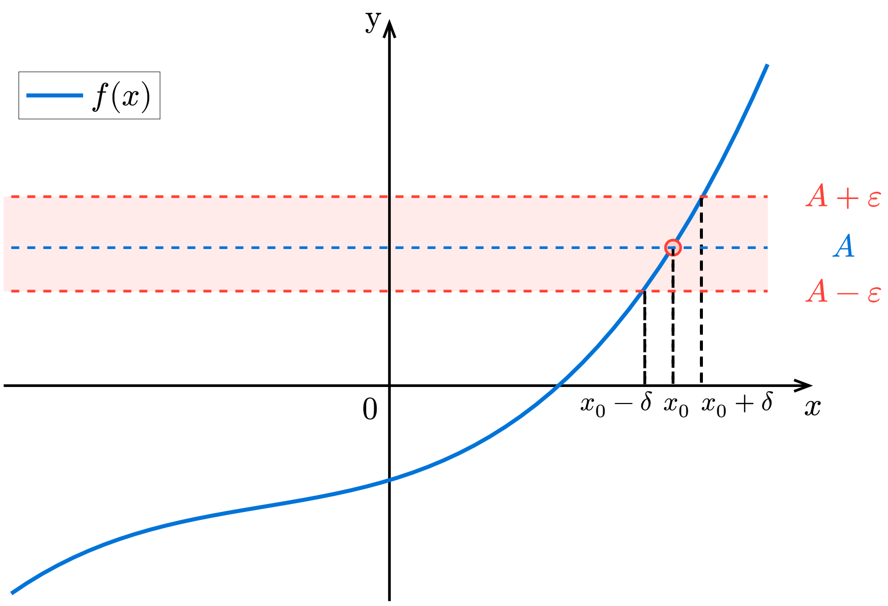
Definition of δ-neighbourhood For any constant $\delta > 0$, we call the open interval $(x_0 - \delta, x_0 + \delta)$ the δ-neighbourhood of x_0.

Now with these definitions, we can easily define a deleted neighbourhood.

Definition of deleted neighbourhood

A δ-neighbourhood of x_0 but without the point x_0.

After these definition, with the aid of the graph down below, we can tear the definition apart scentence by scentence.



We all know that in geometry, the modulus function can mean the distance of two points, so now let's try to interpret

$$|f(x) - A| < \varepsilon$$

as the distance between $f(x)$ and A can be as small as ε, where ε can be as small as we want, $\frac{1}{100000}$, $\frac{1}{1000000000000}$... you name it.

But we don't need the distance of $f(x)$ and A always be this small, we only need this if x is close enough to x_0, so in order to formally express "close enough", we introduced δ and its deleted neighbourhood.

If you still cannot understand what I am saying, you can use the graph up there to understand, now imagine horizontally, there're two lines $y = A + \varepsilon$ and $y = A - \varepsilon$ trying to clasp the line $y = a$. and the same to x-axis on $x - \delta$ and $x + \delta$. 4 lines that clamp (x_0, A) like a sandwich.

Now with the basic understanding of limits, there is another thing that is worth to emphasize.

> ⚠ **A common mistake**
>
> The fact that $\lim_{x \to x_0} f(x)$ exists **has nothing to do** with whether $f(x)$ is defined at x_0 or the value of $f(x_0)$. It is important to know that no matter what situation is in limits. x will never be x_0 but can be infinitely approached to x_0

A perfect example will be this piecewise function.

$$f(x) = \begin{cases} x & (x \neq 0) \\ 1 & (x = 0) \end{cases}$$

What is $\lim_{x \to 0} f(x)$? If you said 1 you're cooked and you should read the content inside warning box again. The correct answer is 0.

1.5. Give it a shot.

After all these yapping I'm sure you wanna give a shot about what you've learned.

Question 1.5.1 Evaluate

$$\lim_{x \to 0} (2x - 1)$$

Solution 1.5.1
Since $2x - 1$ is continous and defined on $x = 0$, we can directly plug in $x = 0$ into $2x - 1$.

Giving 0 - 1 which is 1

This leaves us a very intuitive theorem

Theorem 1.5.1 If a function $f(x)$ is **continous** and defined at x_0, then

$$\lim_{x \to x_0} f(x) = f(x_0)$$

The piecewise function we defined previously is not continous, therefore this theorem cannot apply.

Now let's try something a bit harder

Question 1.5.2 Evaluate

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Solution 1.5.2

Notice that the numerator can be expand to $(x-1)(x+1)$ and $(x-1)$ cancels out with the denominator. Therefore we only need to calculate $\lim_{x \to 1} x + 1$ which according to *Theorem 1.5.1*, the answer is 2.

1.6. Left-hand and right-hand limits

When we define limits previously, we require that x_0 must be the midpoint of the interval according to the definition of neighbourhood, but sometimes we only wanna do a side, when these time come we use left-hand limits and right hand limits.

Definition of right-hand limits

for all $\varepsilon > 0$,there exists a $\delta > 0$ when $x_0 < x < x_0 + \delta$ there is $f(x) - A < \varepsilon$, we call

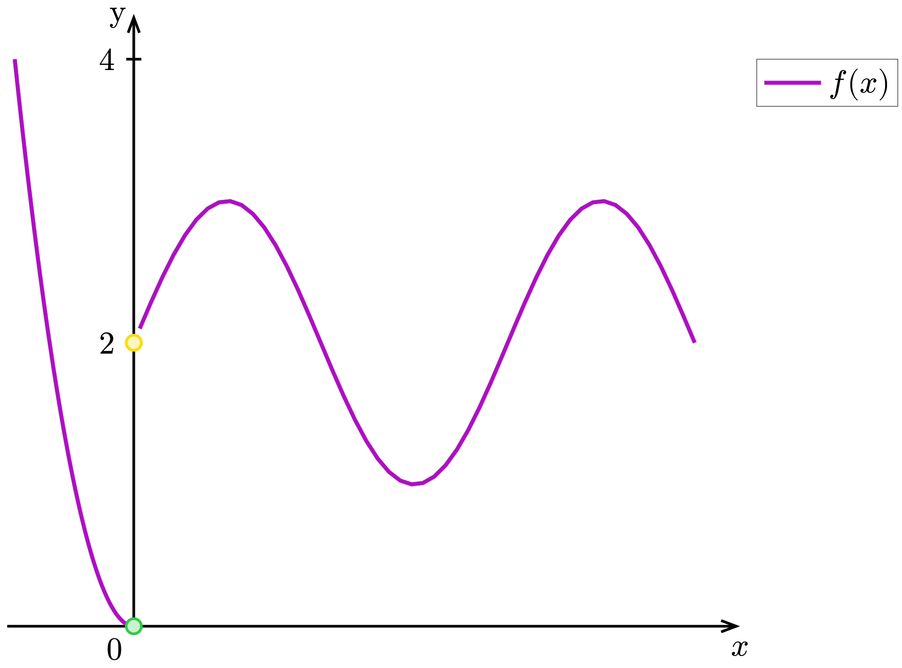
$$\lim_{x \to x_0^+} f(x) = A$$

Definition of left-hand limits

for all $\varepsilon > 0$,there exists a $\delta > 0$ when $x_0 - \delta < x < x_0$ there is $f(x) - A < \varepsilon$, we call

$$\lim_{x \to x_0^-} f(x) = A$$

In order to understand what the skibidi is this,let's take a look at this function.



What is the answer of $\lim_{x \to 0} f(x)$? It doesn't exist.

Why? because $f(x)$ has two corresponding points when $x = 0$, therefore the limit of $f(x)$ when $x = 0$ does not exist.

That's when the left-hand and right-hand limits come in handy, the left-hand limit of $f(x)$ should be 0 and the right-hand limit of $f(x)$ should be 2 according to the graph, it's really intuitive.

This gives us a conclusion.

Theorem 1.6.1

$\lim_{x \to x_0} f(x)$ exists if and only if $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ both exist and equal.

With these in mind, let's look at this question.

Question 1.6.1

let

$$f(x) = \begin{cases} x - 1 \quad (x < 0) \\ 0 \quad (x = 0) \\ x + 1 \quad (x > 1) \end{cases}$$

, prove that $\lim_{x \to 0} f(x)$ doesn't exist

⚠ **Warning**

Do not plug $x = 0$ into the limit, recall that *Theorem 1.5.1* only applies on **continous** function. (I have predicted yall)

Solution 1.6.1

From *Theorem 1.6.1*, we know that if a limit does not exists, the left-hand limit and the right-hand limit of the function must be inequal, with this in mind, we have

$$\lim_{x \to 0^-} f(x) = 0 - 1 = -1$$

$$\lim_{x \to 0^+} f(x) = 0 + 1 = 1$$

left-hand limit and the right-hand limit are inequal,therefore $\lim_{x \to 0} f(x)$ does not exist.

There are also another 2 cases where a function's limit does not exist.

1. This function's left or right hand limit is $\pm\infty$ (Example: $\lim_{x \to 0} \frac{1}{x}$)
2. The function vibrates infinitely (Example: $\lim_{x \to 0} \sin \frac{1}{x}$)

1.7. Limits toward ∞

The limits toward a constant acts differently from limits toward infinity.

Definition of limits toward infinity

let $f(x)$ be defined when $|x|$ is greater than a positive constant, for any $\varepsilon > 0$ (no matter how small it is), there always exists a positive constant X that when $|x| > X$, $|f(x) - A| < \varepsilon$ is always true. Then, we denote that

$$\lim_{x \to \infty} f(x) = A$$

The left-hand and right-hand limits toward infinity is nearly identical to the definition of limits toward infinity, just change $|x| > X$ to $x > X$ and $x < -X$, these limits also satisfy *Theorem 1.6.1*

1.8. L'Hopital's rule

All these things that were mentioned were theroretical, and in A-level Mathematics & Further Mathematics exams, you will mostly use L'Hopital's rule.

L'Hopital's rule

if $\lim_{x \to a} f(x) = \infty$ or 0, and $\lim_{x \to a} g(x) = \infty$ or 0, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

This is a very useful rule since most polynomials and other kinds of functions you meet in the exam, such as *Question 1.5.2* can be applied to this rule.

The questions that uses L'Hopital's rule will usually appear in three forms, they are

1. $\frac{0}{0}$ form
2. $\frac{\infty}{\infty}$ form
3. $0 \cdot \infty$ form

Question 1.8.1 ($\frac{0}{0}$ form)

Use L'hopital's rule to verify that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Caution

Notice that I use **verify** instead of **prove**, this is because the proof of L'Hopital's rule uses this result, proving the statement with L'Hopital's rule will cause a circular reasoning°.

Solution 1.8.1

Notice that $\lim_{x \to 0} \sin x = 0$ and $\lim_{x \to 0} x = 0$,this is why it's called a $\frac{0}{0}$ form. The other two are the same.

By using L'Hopital's rule, we can get that the limit is equal to

$$\lim_{x \to 0} \cos x$$

Since $\cos x$ is continous and defined on $x = 0$, we can directly plug in and get the result $\cos 0 = 1$

Question 1.8.2 ($\frac{\infty}{\infty}$ form)

Prove that

$$\lim_{x \to \infty} \frac{e^x - e^{-x}}{x^2} = \infty$$

Note

You might have the question that why a limit of a function can be ∞, you can see the cases where a function's limit do not exist in the above parts, that doesn't confine that the function's limit itself cannot be ∞, it just ensures that the left and right hand limits cannot be $\pm\infty$. Don't get confused.

Solution 1.8.2

Using L'Hopital's rule we know that the limit is equal to

$$\lim_{x\to\infty} \frac{e^x + e^{-x}}{2x}$$

Apply the L'Hopital rule again. We get

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{2}$$

According to *Theorem 1.5.1* , we plug in $x = \infty$, therefore the answer is ∞

Question 1.8.3 ($0 \cdot \infty$ form)

Find the value of

$$\lim_{x\to 0} x \ln x$$

Solution 1.8.3

This needs a bit of thinking because L'Hopital's rule can be only applied with division than product, but we can change the form of the limit to

$$\lim_{x\to 0} \frac{\ln x}{x^{-1}}$$

Now we can apply the L'Hopital rule, so this limit is equal to

$$\lim_{x\to 0} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0}(-x) = 0$$

Question 1.8.4 (Exercise)

In this problem,you may assume all the function mentioned are continous and logarithms can swap places with limits.

(i) By using L'hopital rule and logarithms, prove that

$$\lim_{f(x)\to 0}(1+f(x))^{\frac{1}{f(x)}} = e$$

(ii) Hence, compute

$$\lim_{x\to 0}\cos 2x^{\frac{3}{x^2}}$$

2. Extended Calculus Technique

> The world of calculus is magnificant except I can do 0 questions in exam. —— Justin Yu

Well, for this chapter I mainly wanna cover the techniques of elementary techniques that won't be taught in school but very helpful.

You probably will find differentiation techniques is extremely short compared to the integration technique. Why will this happen?

2.1. Why integration is just harder

Well the answer is actually quite simple. Differentiation is the process of eliminating the information, and integration is to generate the information from thin air. Imagine you have a piece of artwork, and you throw it into the shredder, this is the process of differentiation, if you have thousands of pieces of paper and you wanna put them together into that same piece of artwork, this is the process of integration. It is obvious that putting back the pieces together is significantly harder than just throw it into the shredder.

Okay quit yapping,now here's something real.

2.2. Differentiation techniques

There's nothing much about actual application of differentiation in A-level Mathematics and Further Mathematics other than what you have been told in the textbook,but I still managed to find some easy and useful techniques that you might find useful.

2.2.1. Logarithmic differentiation

This technique is extremely useful when it comes to differentiating horrendous polynomials, let's say we wanna find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{6(1+2t^2)(t^3-t)^2}{\sqrt{t+5t^2}\cdot 4t^{\frac{3}{2}}}\right)$$

This will took ages to find if you apply the rules and chain rules you've learned with brute force, but with Logarithmic differentiation,the reult would be much easier to get, but before that, I wanna talk about why we want logarithms to involve in when we differentiating a function.

We all know that when $f(x) = f(a) \pm f(b)$, $f'(x) = f'(a) \pm f'(b)$. This rule is really simple and intuitive, but doing products and division is not that beautiful at all. This is why we introduced logarithms here,because logarithms has two rules that we love (I'll use ln here because the base doesn't matter)

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

Did you discover something? logarithms turn the product and division which we dislike into simple addition and subtraction. This is why we want logarithms.

With this in mind, the formula of logarithmic differentiation is extremely easy to discover.

We have learned in A-level Mathematics that

$$[\ln(f(x))]' = \frac{f'(x)}{f(x)}$$

We isolate $f'(x)$ to one side to get

$$f'(x) = f(x) \cdot [\ln(f(x))]'$$

This is the formula of logarithmic differentiation, you don't need to recite the formula itself because the mindset behind it can let you solve the questions I mentioned above without this formula.

The logarithmic differentation formula

$$f'(x) = f(x) \cdot [\ln(f(x))]'$$

Now let's give a try about what you've just saw.

Question 1

Differentiating

$$\frac{\sqrt[5]{x-3} \cdot \sqrt[3]{3x-2}}{\sqrt{x+2}}$$

Solution 1

We let

$$y = \frac{\sqrt[5]{x-3} \cdot \sqrt[3]{3x-2}}{\sqrt{x+2}}$$

by taking logarithms from both sides, we get

$$\ln y = \frac{1}{5}\ln|x-3| + \frac{1}{3}\ln|3x-2| - \frac{1}{2}\ln|x+2|$$

Differentiating both sides,

$$\frac{y'}{y} = \frac{1}{5(x-3)} + \frac{1}{3(3x-2)} - \frac{1}{2(x+2)}$$

Therefore

$$y' = \frac{\sqrt[5]{x-3} \cdot \sqrt[3]{3x-2}}{\sqrt{x+2}} \cdot \left(\frac{1}{5(x-3)} + \frac{1}{3(3x-2)} - \frac{1}{2(x+2)}\right)$$

Some of you might wonder why I add absolute value to $x-3$, **this is because we require $x > 0$ inside $\ln(x)$**

This is extremely important,otherwise your function could be in trouble because of domain.

Question 2

Let

$$f(x) = \prod_{a=1}^{114514} (e^{ax} - a)$$

, compute $f'(0)$

Solution 2

First we evaluate $f'(x)$ by formula,which is

$$f'(x) = \prod_{a=1}^{114514} (e^{ax} - a) \cdot \sum_{b=1}^{114514} \frac{be^{bx}}{e^{bx} - b}$$

The key here is: when $x = 0$,the first term will appear $\frac{1}{0}$,which is obviously not what we want.

To solve this,we isolate the first term inside the sum.

$$f'(x) = \prod_{a=1}^{114514} (e^{ax} - a) \cdot \frac{e^x}{e^x - 1} + \prod_{a=1}^{114514} (e^{ax} - a) \cdot \sum_{b=2}^{114514} \frac{be^{bx}}{e^{bx} - b}$$

And we bring out the first term in the product to cancel out $e^x - 1$

$$f'(x) = \prod_{a=2}^{114514} (e^{ax} - a) \cdot \frac{(e^x - 1)e^x}{e^x - 1} + \prod_{a=1}^{114514} (e^{ax} - a) \cdot \sum_{b=2}^{114514} \frac{be^{bx}}{e^{bx} - b}$$

Simplify,we get

$$f'(x) = e^x \cdot \left(\prod_{a=2}^{114514} (e^{ax} - a)\right) + \prod_{a=1}^{114514} (e^{ax} - a) \cdot \sum_{b=2}^{114514} \frac{be^{bx}}{e^{bx} - b}$$

Plug in $x = 0$,for the second term,since $e^0 - 1 = 0$,the entire term will be 0, for the first term.

$$f'(0) = \prod_{a=2}^{114514} (1 - a)$$

which is $(-1)(-2)(-3)....(-114513)$, which equals to $-114513!$

If you're interested in this, you can use the method above to compute $f'(0)$ when

$$f(x) = \prod_{a=1}^{n} (e^{ax} - a)$$

Here are some exercise questions

Question 3-6

3. find the derivative of

$$f(x) = \frac{x^2}{1 - x} \sqrt[3]{\frac{3 - x}{(3 + x)^2}}$$

4. find $f'(0)$ when

$$f(x) = \frac{6x - x^3 + 5x^4 + e^x}{5 + 3x^2 + 2x^3 + \cos(x)}$$

5. find the derivative of

$$f(x) = \prod_{i=1}^{n} (x - a_i)^{a_i}$$

6. let $f(x)$ and $g(x)$ be differentatiable functions of x,find the derivative of

$$y = \log_{f(x)} g(x)$$

2.2.2. Some notes about the domain problem

Some of you might wonder why the absolute sometimes appear and sometimes not in the questions we just demonstrate, here I wanna state that this method can only produce numerically correct result, if you want a very strict process, you need to consider the limit when x approaches to 0

2.3. Calculus techniques

The calculus techniques will be divided into 2 sections, indefinite integrals and definite integrals.

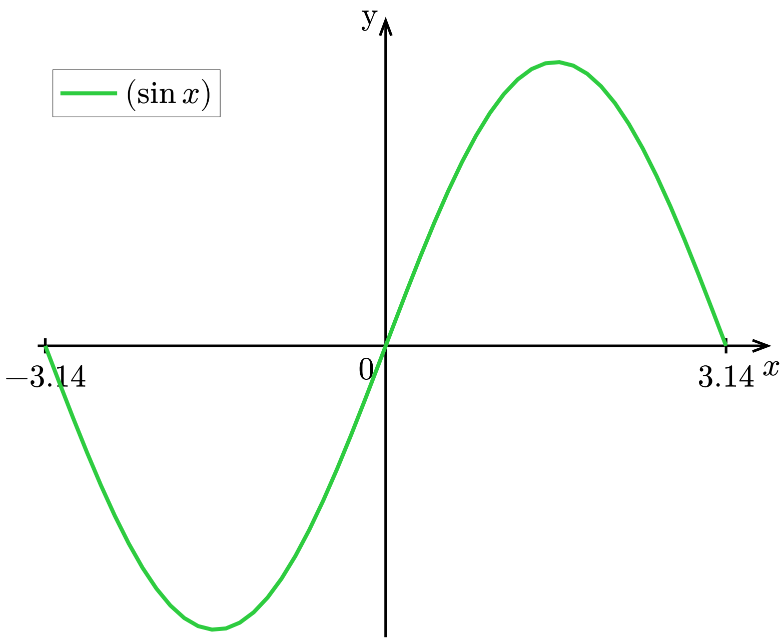
> **Note**
>
> The technique mentioned here are within integral of a single-variable function. Techniques beyond this such as Feynman's integration technique or residue calculus will not be appeared in this section.

2.3.1. Definite integral techniques

2.3.1.1. Definite integrals of odd/even functions

First let's do a quick recap about what is even and odd function.

Odd function are function $f(x)$ such that $f(-x) = -f(x)$, $\sin(x)$ is a very classic odd function, it is origin symmetry.



Now, if you have basic math knowledge by knowing what does origin symmetry mean or , you will notice that the surface area under $[-\pi, 0]$ and $[0, \pi]$ are same, and by using the geometry definition of definite integral,we get

$$\int_{-\pi}^{\pi} \sin(x)\ \mathrm{d}x = 0$$

.

Of course since the function is origin symmetry, we can easily get the theorem shown below.

Theorem 2.3.1

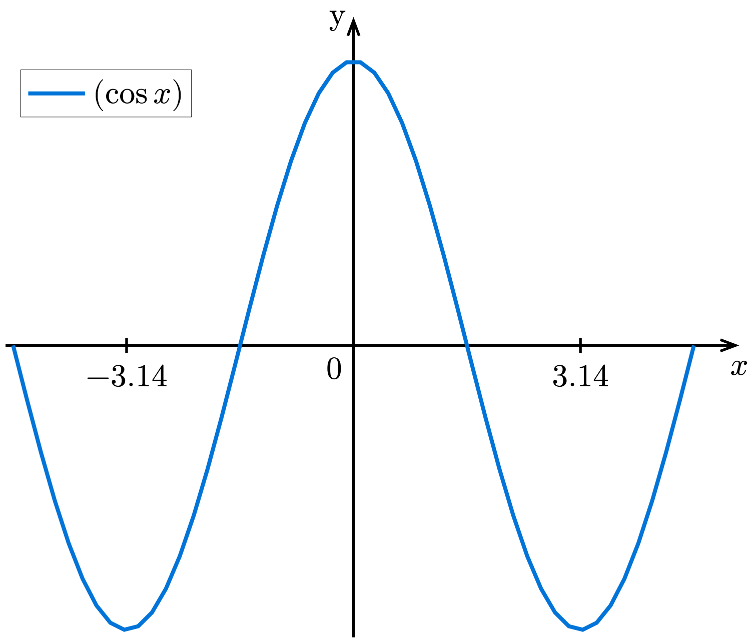
For any continous odd function $f(x)$,

$$\int_{-a}^{a} f(x)\ \mathrm{d}x = 0$$

where a and $-a$ are within the domain of $f(x)$.

Well smart you must have guessed that there's a same thing for even function, but before that let's simply introduce it.

Even function are function such that $f(-x) = f(x)$, $\cos(x)$ is a classic even function, even functions are symmetry to y-axis.



It is also extremely easy to understand that

Theorem 2.3.2

For any continous even function $f(x)$,

$$\int_{-a}^{a} f(x)\ \mathrm{d}x = 2\int_{0}^{a} f(x)\ \mathrm{d}x$$

where a and $-a$ are within the domain of $f(x)$.

This is all inferred by the theorem below

Theorem 2.3.3

For **any** integratable continous function $f(x)$, we have

$$\int_{-a}^{a} f(x)\ \mathrm{d}x = \int_{0}^{a} [f(x) + f(-x)]\ \mathrm{d}x$$

where a and $-a$ are within the domain of $f(x)$.

Proof.

Obviously,

$$\int_{-a}^{a} f(x)\ \mathrm{d}x = \int_{-a}^{0} f(x)\ \mathrm{d}x + \int_{0}^{a} f(x)\ \mathrm{d}x$$

let $u = -x$, $\mathrm{d}u = -\mathrm{d}x$

$$\int_{-a}^{a} f(x)\ \mathrm{d}x = \int_{a}^{0} f(-u)\ \mathrm{d}(-u) + \int_{0}^{a} f(x)\ \mathrm{d}x$$

(definite integral is required to change the upper bound and lower bound while using substitution)

$$\int_{-a}^{a} f(x)\ \mathrm{d}x = -\int_{0}^{a} f(-u)\ \mathrm{d}u + \int_{0}^{a} f(x)\ \mathrm{d}x$$

by recalling -$\int_{b}^{a} f(x)\ \mathrm{d}x = \int_{a}^{b} f(x) \mathrm{d}x$

$$\int_{-a}^{a} f(x)\ \mathrm{d}x = \int_{0}^{a} f(-u)\ \mathrm{d}u + \int_{0}^{a} f(x)\ \mathrm{d}x$$

$$\int_{-a}^{a} f(x)\ \mathrm{d}x = \int_{0}^{a} f(-x)\ \mathrm{d}x + \int_{0}^{a} f(x)\ \mathrm{d}x$$

$$\int_{-a}^{a} f(x)\ \mathrm{d}x = \int_{0}^{a} [f(x) + f(-x)]\ \mathrm{d}x$$

□

The form where the upper bound and the lower bound of the integral is actually very easy to observe, the hard part of this is to verify whether the integrated function is an odd/even function.

Let's look at some example questions

Question 7

Compute

$$\int_{-2}^{2} x \ln(1 + e^x)\ \mathrm{d}x$$

Solution 7

By using *Theorem 2.3.3*, we can transform the original integral into

$$\int_{0}^{2} x \ln\left(\frac{1 + e^x}{1 + e^{-x}}\right)\ \mathrm{d}x$$

we times e^x to the upper and lower part of the fraction, it is

$$\int_{0}^{2} x \ln\left(\frac{e^x \cdot (e^x + 1)}{e^x + 1}\right)\ \mathrm{d}x$$

This is just

$$\int_{0}^{2} x^2\ \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3}$$

Question 8

Compute

$$\int_{-1}^{1} \frac{2x^2 + \sin x}{1 + \sqrt{1 - x^2}}\ \mathrm{d}x$$

Solution 8

At first glance the function is not an even nor an odd function, applying *Theorem 2.3.3*, seems a bit complicated as well. Trying to decomposite the function so we might get some clue. The original integral is equal to

$$\int_{-1}^{1} \frac{2x^2}{1 + \sqrt{1 - x^2}}\ \mathrm{d}x + \int_{-1}^{1} \frac{\sin x}{1 + \sqrt{1 - x^2}}\ \mathrm{d}x$$

Here we're! the first term is an even function and the second term is an odd function, so the second term directly equals to 0, this gives us

$$4\int_0^1 \frac{x^2}{1+\sqrt{1-x^2}}\,\mathrm{d}x$$

So how do we solve this?, we can try make the denominator rational, so times by $1-\sqrt{1-x^2}$, the form below is a^2-b^2 , which after simplifying, looks like this

$$4\int_0^1 \frac{x^2\left(1+\sqrt{1-x^2}\right)}{1-1+x^2}\,\mathrm{d}x = 4\int_0^1 \left(1-\sqrt{1-x^2}\right)\,\mathrm{d}x$$

this is

$$4-4\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$$

Ok , you can use the substitution $u=\sin(\theta)$ to solve the integration, but this obviously would take some time, let's try to use some geometry here.

let $y=\sqrt{1-x^2}$, we can get $x^2+y^2=1$, which is a unit circle. integrate this from 0 to one is actually a quarter of the whole circle, which is just $\frac{\pi}{4}$, so the answer is

$$4-4\cdot\frac{\pi}{4}=4-\pi$$

Question 9

let $g(t)$ be a contionous odd function, compute

$$\int_{-\frac{2}{\pi}}^{\frac{2}{\pi}} \left[\int_0^x g(t)\ln\left(\frac{1-t}{1+t}\right)\mathrm{d}t + \frac{x^2\cos x}{1+e^x}\right]\,\mathrm{d}x$$

Source: JEE-Mains 2024 Shift 2 QP 27-Jan

Solution 9

First rule when you see multiple integrals in an highschool math exam is DO NOT PANIC. because at 99% of the time you can get rid of them by using the property of odd function.

So the question gives us that $g(x)$ is an odd function, we wanna get rid of the integration, then $\int_0^x g(t)\ln\left(\frac{1-t}{1+t}\right)\mathrm{d}t$ must be an odd function. Let's see why it is first.

We let $f(x)=\int_0^x g(t)\ln\left(\frac{1-t}{1+t}\right)\mathrm{d}t$ then,

$$f(-x)=\int_0^{-x} g(t)\ln\left(\frac{1-t}{1+t}\right)\,\mathrm{d}t$$

Considering substitution $u=-t$

$$f(-x) = \int_0^x g(u) \ln\left(\frac{1+u}{1-u}\right) \mathrm{d}u$$

By the laws of logarithms $\ln\left(\frac{1+u}{1-u}\right) = -\ln\left(\frac{1-u}{1+u}\right)$, so

$$f(-x) = -\int_0^x g(u) \ln\left(\frac{1-u}{1+u}\right) \mathrm{d}u$$

Obviously $\int_0^x g(u) \ln\left(\frac{1-u}{1+u}\right) \mathrm{d}u = \int_0^x g(t) \ln\left(\frac{1-t}{1+t}\right) \mathrm{d}t = f(x)$, we bring the substitution back. It is

$$f(-x) = -f(x)$$

With this proof, all we have left is this integration

$$g(x) = \int_{-\frac{2}{\pi}}^{\frac{2}{\pi}} \frac{x^2 \cos x}{1+e^x} \mathrm{d}x$$

the numerator is an odd function, but the denominator isn't an even nor an odd function.

let's try substitution $u = -x$ first to see if we can use *Theorem 2.3.3*, after substitution it is

$$g(-x) = \int_{\frac{2}{\pi}}^{-\frac{2}{\pi}} \frac{u^2 \cos u}{1+e^{-u}} \mathrm{d}(-u) = \int_{-\frac{2}{\pi}}^{\frac{2}{\pi}} \frac{u^2 \cos u}{1+e^{-u}} \mathrm{d}u$$

let's time e^u On both sides. which is

$$g(-x) = \int_{-\frac{2}{\pi}}^{\frac{2}{\pi}} \frac{e^u u^2 \cos u}{1+e^u} \mathrm{d}u$$

The original integration has $u^2 \cos u$, adding these two together will let the denominator cancel out.

$$\int_{-\frac{2}{\pi}}^{\frac{2}{\pi}} g(x) = \frac{1}{2} \int_{-\frac{2}{\pi}}^{\frac{2}{\pi}} [g(x) + g(-x)] = \int_0^{\frac{2}{\pi}} x^2 \cos x \ \mathrm{d}x = \frac{\pi^2}{4} - 2$$

Here are some questions for exercise

Question 10-13

10. Compute

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x - \cos^3 x} \ \mathrm{d}x$$

11. By considering substitution $t = x - \frac{\pi}{2}$, compute

$$\int_0^{\pi} \frac{\pi + \cos x}{x^2 - \pi x + 2026} \, \mathrm{d}x$$

12. Compute

$$\int_{-1}^{1} \sin^2 x \cdot \ln\left(x + \sqrt{1 + x^2}\right) \, \mathrm{d}x$$

13. Compute

$$\int_{-3}^{3} x \cos x + 2\sqrt{9 - x^2} \, \mathrm{d}x$$

2.3.1.2. Defnite integrals of periodic functions

Periodic functions are functions that replicate itself when reach a certain period T, for any periodic functions $f(x)$, $f(x + T) = f(x)$.

For example, the period of $\sin x$ is 2π. We got 2 theorem for periodic functions.

Theorem 2.3.4

For any periodic function $f(x)$ with period T,

$$\int_a^{a+T} f(x) \, \mathrm{d}x = \int_0^T f(x) \, \mathrm{d}x$$

If you use the area under curve to understand this theorem it would be an ease, but I'm still going to provide an algebraically proof below.

Proof.

$$\int_a^{a+T} f(x) \, \mathrm{d}x = \int_a^0 f(x) \, \mathrm{d}x + \int_0^T f(x) \, \mathrm{d}x + \int_T^{a+T} f(x) \, \mathrm{d}x$$

Considering substitution $u = x - T$

$$\int_T^{a+T} f(x) \, \mathrm{d}x = \int_0^a f(u + T) \, \mathrm{d}u = \int_0^a f(u) \, \mathrm{d}u = \int_0^a f(x) \, \mathrm{d}x$$

So the original equation can be rewritten into

$$\int_a^{a+T} f(x) \, \mathrm{d}x = \int_a^0 f(x) \, \mathrm{d}x + \int_0^T f(x) \, \mathrm{d}x + \int_0^a f(x) \, \mathrm{d}x = \int_0^T f(x) \, \mathrm{d}x$$

□

Here's another one

Theorem 2.3.5

For any periodic function $f(x)$ with period T,

$$\int_0^{nT} f(x)\ \mathrm{d}x = n \int_0^{T} f(x)\ \mathrm{d}x$$

With *Theorem 2.3.4* proving this will be easy, so I'll not talk about much in here.

Question 14

Compute

$$\int_0^{6\pi} \frac{\sin x}{5 + 3\sin^2 x}\ \mathrm{d}x$$

Solution 14

notice that $\sin(x)$ appear on both sides of the fraction, we guess that the period of this function is related to π, and since $\sin(x + 2\pi) = \sin(x)$, the period of this function is also 2π. Also this function is an odd function, obviously.

the original integral is just

$$3 \int_0^{2\pi} \frac{\sin x}{5 + 3\sin^2 x}\ \mathrm{d}x$$

Use *Theroem 2.3.4* reversely, let a be $-\pi$ to construct a symmetry upper and lower bound. This gives us

$$3 \int_{-\pi}^{\pi} \frac{\sin x}{5 + 3\sin^2 x}\ \mathrm{d}x$$

Since the function inside is just an odd function, the answer to the integral is 0

Question 15

Compute

$$\int_0^{2\pi} |\sin(x + 1)|\ \mathrm{d}x$$

Solution 15

The period of this function is obviously π by looking at the graph.

First we let $u = x + 1$ to get a clear view inside sine function.

$$\int_1^{2\pi+1} |\sin(u)|\ \mathrm{d}u = \int_0^{2\pi} |\sin(u)|\ \mathrm{d}u = 2\int_0^{\pi} \sin(u)\ \mathrm{d}u$$

(still by using *Theroem 2.3.4* reversely, set a to 0, and we get rid of the absolute value because the given boundaries are always positive)

It is very easy to aquire the result, which is 4.

Here below are some exercise questions

Question 16-18

16. Compute

$$\int_0^{2025\pi} \sqrt{1 - \cos 2x}\ \mathrm{d}x$$

17. Compute

$$\int_0^{100\pi} \left(\frac{x}{\pi} - \left\lfloor \frac{x}{\pi} \right\rfloor\right) \cdot \frac{|\sin x|}{1 + \cos^2 x}\ \mathrm{d}x$$

Here $\lfloor x \rfloor$ denotes the gauss floor function ($\lfloor 1.732 \rfloor = 1$)

18. Compute

$$\int_{e^{-2n\pi}}^{1} \left| \frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\left(\ln \frac{1}{x}\right)\right) \right|\ \mathrm{d}x$$

where $n \in \mathbb{N}$

2.3.1.3. King's rule

Well King's rule is badly named in my opnion, The chinese word for the king's rule is "interval reapperance". To be honest I like this much more because it actually explains the technique.

The King's rule is just a simple formula.

The King's Rule

For any integratable continous function $f(x)$

$$\int_b^a f(x)\ \mathrm{d}x = \int_b^a f(a+b-x)\ \mathrm{d}x = \frac{1}{2}\int_b^a [f(x)+f(a+b-x)]\ \mathrm{d}x$$

Proof. Considering substitution $u = a+b-x$, $x = a+b-u$, $\mathrm{d}u = -\mathrm{d}x$

$$\int_b^a f(x)\ \mathrm{d}x = -\int_a^b f(a+b-u)\ \mathrm{d}u$$

(Notice the change of upper and lower bound of the integral)

This gives us

$$\int_b^a f(a+b-u)\ \mathrm{d}u = \int_b^a f(a+b-x)\ \mathrm{d}x$$

□

With the King's rule we can infer this theorem.

Theorem 2.3.6

If function $f(x), g(x)$ satisfy $f(x) = f(a+b-x), g(x)+g(a+b-x) = m$ (m is an constant), then

$$\int_b^a f(x)g(x)\ \mathrm{d}x = \frac{m}{2}\int_b^a f(x)\ \mathrm{d}x$$

Proof.

$$I = \int_b^a f(x)g(x)\ \mathrm{d}x = \int_b^a f(a+b-x)g(a+b-x)\ \mathrm{d}x$$

Which is

$$\int_b^a f(x)g(a+b-x)$$

(Apply $f(x) = f(a+b-x)$)

so

$$2I = \int_b^a f(x)g(x)\ \mathrm{d}x + \int_b^a f(x)g(a+b-x)\ \mathrm{d}x = \int_b^a f(x)[g(x)+g(a+b-x)]\ \mathrm{d}x$$

Because $g(x)+g(a+b-x) = m$

$$2I = m\int_b^a f(x)\ \mathrm{d}x$$

$$I = \frac{m}{2}\int_b^a f(x)\ \mathrm{d}x$$

□

Do not underestimate the power of these rule, with these rule you can get extremely difficult questions in the actual work.

Question 19

Compute

$$\int_0^{\frac{\pi}{2}} x\sin^2 x\cos^2 x\ \mathrm{d}x$$

Solution 19

By using double angle formula, $(\sin x\cos x)^2 = \frac{1}{4}\sin^2 2x$, so this gives us

$$\frac{1}{4}\int_0^{\frac{\pi}{2}} x\sin^2 2x\ \mathrm{d}x$$

By using substitution $u = 2x$ this is equal to

$$\frac{1}{16}\int_0^{\pi} u\sin^2 u\ \mathrm{d}u$$

Considering using *Theorem 2.3.6*,by setting $f(x) = \sin^2 x, g(x) = x, g(x) + g(0 + \pi - x) = x + \pi - x = \pi$. Therefore the integral is equal to

$$\frac{\pi}{32}\int_0^{\pi} \sin^2 u\ \mathrm{d}u = \frac{\pi}{32}\cdot\frac{\pi}{2} = \frac{\pi^2}{64}$$

Question 20

Compute

$$\int_0^{\pi} \frac{x|\sin x\cos x|}{1+\cos^2 x}$$

Solution 20

By the King's rule, we consider substitution $t = \pi - x$, where $x = \pi - t$, $\mathrm{d}x = -\mathrm{d}t$, this gives us

$$\int_0^\pi \frac{(\pi - t)|\sin(\pi - t)\cos(\pi - t)|}{1 + \cos^2(\pi - t)}\ \mathrm{d}t = \int_\pi^0 \frac{(\pi - t)|\sin t \cos t|}{1 + \cos^2 t}\ \mathrm{d}(-t)$$

$$= \int_0^\pi \frac{(\pi - x)|\sin x \cos x|}{1 + \cos^2 x}\ \mathrm{d}x$$

($d(-t)$ is used to reverse the upper and lower bound of the integral)

Now, we add this integration and the original integration to cancel out x (*Theorem 2.3.3*), we get

$$\frac{\pi}{2}\int_0^\pi \frac{|\sin x \cos x|}{1 + \cos^2 x}\ \mathrm{d}x$$

Now with the absolute value and the sqaure, we're pretty sure that this function is an even function, so we construct a symmetry upper and lower bound, we can do this because this function is symmetry from y-axis and it has a period of π.

$$\frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{|\sin x \cos x|}{1 + \cos^2 x}\ \mathrm{d}x = \pi \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{1 + \cos^2 x}\ \mathrm{d}x$$

(Remove the absolute value because the interval is positive)

Notice that $\left(\cos^2 x\right)' = -2\sin x \cos x$, so by substitution $u = \cos^2 x$ we can get

$$-\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1 + u}\ \mathrm{d}u = -\frac{\pi}{2}\left[\ln\left(1 + \cos^2 x\right)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}\ln 2$$

Here are some exercise questions

Question 20-30

20. Prove that for any even function $f(x)$.

$$\int_{-a}^{a}\int_0^x f(t)\ \mathrm{d}t\ \mathrm{d}x = 0$$

21. Compute

$$\int_{-\pi}^{\pi} \frac{x \sin x \left[\tan^{-1}(e^x) + \int_0^x \left(\frac{4t^6 \ln(1+t^2)\cos(t^4+1)}{\sin^2 t + t^2 + e^{-2t^2}} + \tan^{-1}\left(\frac{e^{3t} + e^{-3t}}{2}\right)\right)\mathrm{d}t\right]}{1 + \cos^2 x}\ \mathrm{d}x$$

22. Compute

$$\int_0^1 \frac{(2x-1)^2 \tan^{-1}\left(\frac{1}{x-1}\right)}{(x^2-x+4)\sqrt{x^2-x+\frac{1}{4}}}\ \mathrm{d}x$$

23. Compute

$$\int_0^{\frac{\pi}{4}} \ln\left(\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}t}{1+\tan^{\alpha} t} + \tan x\right)\ \mathrm{d}x$$

24. Compute

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^{-1}(e^x)\sin^2 x\ \mathrm{d}x$$

25. Compute

$$\int_0^1 \frac{\sin^{-1}(\sqrt{x})}{\sqrt{x^2-x+1}}\ \mathrm{d}x$$

26. By considering substitution $t = \sin x$, compute

$$\int_0^1 \frac{\sin^{-1}(x)}{x}\ \mathrm{d}x$$

27. By considering substitution $t = \frac{1}{x}$, compute

$$\int_{\frac{1}{2}}^2 \frac{\tan^{-1}(x)}{x^2+x+1}\ \mathrm{d}x$$

28. By considering substitution $u = \tan x$. compute coxeter's integral

$$\int_0^{\frac{\pi}{4}} \tan^{-1}\left(\sqrt{\frac{\cos 2x}{2\cos^2 x}}\right)\ \mathrm{d}x$$

29.

(i) Compute

$$\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin x + \cos x}\ \mathrm{d}x$$

(ii) Compute

$$\int_{-\pi}^{\pi} |x|(x^3+\sin^2 x)\cos^2 x\ \mathrm{d}x$$

30. Compute

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \tan x} \, \mathrm{d}x$$

2.3.1.4. Wallis's formula

Wallis's formula is a formula sepcifically targeted for the integral of high-power trignometry function.

Wallis's formula

$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x = \begin{cases} \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{n-3}{n-2} \cdot \frac{n-1}{n} & \text{n is odd} \\ \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{n-3}{n-2} \cdot \frac{n-1}{n} & \text{n is even} \end{cases}$$

Proof.

First, let's try prove $\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x$, considering using King's rule, by substituting $x = \frac{\pi}{2} - t, t = \frac{\pi}{2} - x, \mathrm{d}t = -\mathrm{d}x$

$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x = -\int_{\frac{\pi}{2}}^0 \cos^n t \, \mathrm{d}t = \int_0^{\frac{\pi}{2}} \cos^n t \, \mathrm{d}t$$

Hence Q.E.D for the first half of equation.

Then we let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x$, considering substitution $u = \sin x, \mathrm{d}u = \cos x \cdot \mathrm{d}x$

$$I_n = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \, \mathrm{d}u$$

Using integration by parts, we have

$$I_n = [\cos^{n-1} x \cdot \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, \mathrm{d}(\cos^{n-1} x)$$

left side is just 0, $\mathrm{d}(\cos^{n-1} x) = -\sin x \cdot (n-1) \cos^{n-2} x$, therefore.

$$I_n = (n-1) \int_0^{\frac{\pi}{2}} \sin^2 x \cos^{n-2} x \, \mathrm{d}x$$

By using identity $\sin^2 x = 1 - \cos^2 x$, and expand

$$I_n = (n-1) \left(\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, \mathrm{d}x - \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x \right) = (n-1)(I_{n-2} - I_n)$$

By rearranging both sides, we get the recursive formula

$$I_n = \frac{n-1}{n} I_{n-2}$$

When n is odd

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ldots \cdot \frac{2}{3} \cdot I_1$$

where $I_1 = \int_0^{\frac{\pi}{2}} \cos x \, \mathrm{d}x = 1$

When n is even

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ldots \cdot \frac{1}{2} \cdot I_0$$

where $I_0 = \int_0^{\frac{\pi}{2}} 1 \, \mathrm{d}x = \frac{\pi}{2}$

□

Of course you may not seen that the upper bound of the definite integral is actually $\frac{\pi}{2}$ quite often, usually the upper bound will be π or 2π, we can use the graph of $\sin x, \cos x$ for $\sin^n x, \cos^n x$ when n is odd, and $|\sin x|, |\cos x|$ for n is even. **Because these two functions behave the same when they comes to even/odd and periodicity**.

Question 31

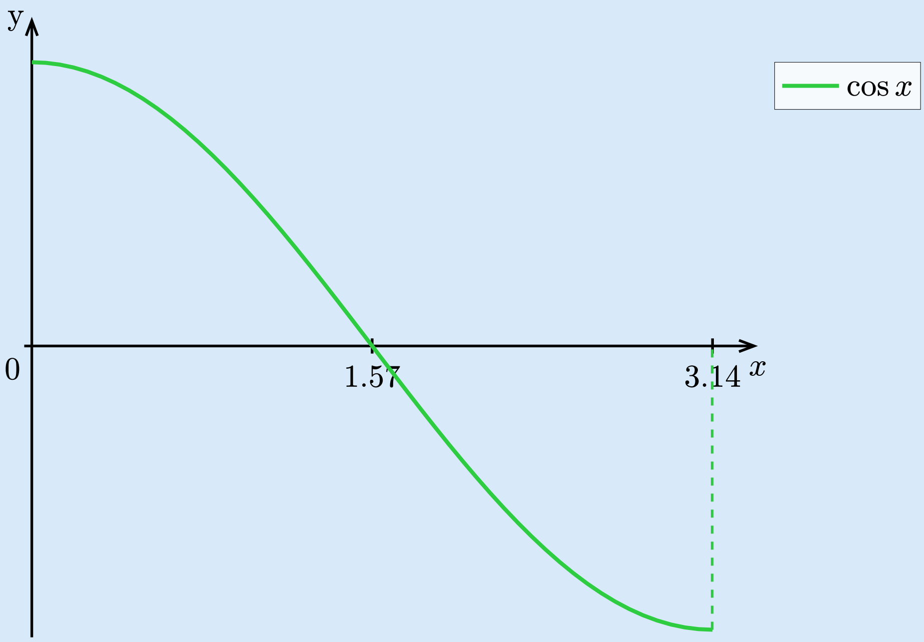
Compute

$$\int_0^{\pi} \cos^{2k-1} x \, \mathrm{d}x$$

where $k \in \mathbb{Z}^+$

Solution 31

You may learn that when $k \in \mathbb{Z}^+, 2k-1$ must be an odd number, so this function behaves the same as $\cos x$, we can draw the diagram of $\cos x$ from 0 to π



notice that the area under curve cancels out, so the result is simply 0.

With this mindset, you can try concluding the formula of $\int_0^\pi \cos^n x \, \mathrm{d}x$, $\int_0^\pi \sin^n x \, \mathrm{d}x$, $\int_0^{2\pi} \sin^n x \, \mathrm{d}x$ and $\int_0^{2\pi} \cos^n x \, \mathrm{d}x$ when n is even or odd.

Question 32

Compute

$$\int_0^{\frac{\pi}{4}} \sin^4 x \, \mathrm{d}x$$

Solution 32

Ofc you can use the wacky ass $\sin^2 x = \frac{1-\cos 2x}{2}$, but a true man will still use Wallis's formula.

Notice that

$$\int_0^{\frac{\pi}{4}} \sin^4 x \, \mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^4 x \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin^4 x \, \mathrm{d}x = \frac{3}{16}\pi$$

We let $I = \int_0^{\frac{\pi}{4}} \sin^4 x \, \mathrm{d}x$, $J = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^4 x \, \mathrm{d}x$, so $I + J = \frac{3}{16}\pi$

Now since we have an $I + J$ and we only want I,why not we construct $I - J$?

$$I - J = \int_0^{\frac{\pi}{4}} \sin^4 x \, \mathrm{d}x - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^4 x \, \mathrm{d}x$$

let's make the bound same so we can compute eaiser, considering substitution $u = \frac{\pi}{2} - x, x = \frac{\pi}{2} - u, \mathrm{d}u = -\mathrm{d}x$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^4 x \, \mathrm{d}x = -\int_{\frac{\pi}{4}}^{0} \cos^4 u \, \mathrm{d}u = \int_{0}^{\frac{\pi}{4}} \cos^4 u \, \mathrm{d}u$$

Now

$$I - J = \int_0^{\frac{\pi}{4}} \left(\sin^4 x - \cos^4 x\right) \mathrm{d}x = \int_0^{\frac{\pi}{4}} \left(\sin^2 x - \cos^2 x\right) \mathrm{d}x = -\int_0^{\frac{\pi}{4}} \cos 2x \, \mathrm{d}x = -\frac{1}{2}$$

So this becomes an elementary school problem.

$$\begin{cases} I + J = \frac{3}{16}\pi \\ I - J = -\frac{1}{2} \end{cases}$$

$$I = \frac{3}{32}\pi - \frac{1}{4}$$

Question 33

Compute

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x \, \mathrm{d}x$$

Solution 33

By observing that $\sin^4 x = \left(1 - \cos^2 x\right)^2$, we can expand the original integral into

$$\int_0^{\frac{\pi}{2}} \left(\cos^5 x + \cos^9 x - 2\cos^7 x\right) \mathrm{d}x$$

By each applying Wallis's formula individually, we have the answer $\frac{8}{315}$

More generally, we have

Theorem 2.3.7

$$\int_0^{\frac{\pi}{2}} \cos^m x \cdot \sin^n x \, \mathrm{d}x = \begin{cases} \frac{(m-1)!! \cdot (n-1)!!}{(m+n)!!} \cdot \frac{\pi}{2} & \text{m,n are both even} \\ \frac{(m-1)!! \cdot (n-1)!!}{(m+n)!!} & \text{otherwise} \end{cases}$$

Here

$$n!! = \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n - 2k)$$

for example

$$9!! = 1 \times 3 \times 5 \times 7 \times 9 = 945$$

$$6!! = 2 \times 4 \times 6 = 48$$

The proof involves beta and gamma function which is beyond what we need to know. Therefore I'll now show it here.

Here below are some exercise questions.

Question 34

34. By considering substitution $u = \sin x$, compute

$$\int_0^1 x^3 \sqrt{1 - x^2} \; \mathrm{d}x$$

2.3.2. Indefinite integral techniques

2.3.2.1. A solution to integral of high-power trignometry function

We all have learnt de Movire's theorem, just a quick recap for yall.

de Movire's theorem

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

Since this can be verified easily by using Euler's formula, I won't talk about it too much in here.