Martices 1

- Note: The offical CIE book covers a lot of the stuff that hodder version and probably collins version didn't cover in this chapter, we'll follow CIE book.

A matrix is something that looks like

$$\boldsymbol{A}_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ a_{m1} & \vdots & \vdots & a_{mn} \end{pmatrix}$$

order = row × column

Special Martices

1. zero matrix, denote by $\boldsymbol{O}_{m\times n}$ where

$$\boldsymbol{O}_{m\times n} = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

2. Identity matrix which is a square matrix (matrix such that order is $n \times n$)

$$\boldsymbol{I}_{n} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

the diagonal line all equals to one and the other element all equals to 0.

3.

Additon, Subtraction and Scalar product of a matrix

Consider two random martices $\boldsymbol{A}_{m\times n}$ and $\boldsymbol{B}_{m\times n}$ and the elements inside are denoted by a_{ij}, b_{ij}

$$\boldsymbol{A} + \boldsymbol{B} = \left(a_{ij} + b_{ij}\right)$$

e.g

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 7 & 9 \end{pmatrix}$$

we can trivially infer that

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$$

$$\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$$

$$\boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$$

for any matrix $\boldsymbol{A}$ there exists a matrix $\boldsymbol{N}$ such that

$$\boldsymbol{A} + \boldsymbol{N} = \boldsymbol{O}$$

consider a scalar λ we define that

$$\lambda\boldsymbol{A} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots \\ \lambda a_{21} & \lambda a_{22} & \cdots \\ \cdots & \cdots & \ddots \end{pmatrix}$$

for subtraction we can infer that

$$\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-1)\boldsymbol{B}$$

the rules are same.

Product of martices

Consider two martices $\boldsymbol{A}_{m\times p} = (a_{ik})$ and $\boldsymbol{B}_{p\times n} = (b_{kj})$

And

$$\boldsymbol{C}_{m\times n} = \boldsymbol{A} \times \boldsymbol{B} = (c_{ij})$$

Here.

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Notice that only the equal column of the first matrix and the row of the second matrix produces a valid product.

e.g

$$\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \text{row1} \times \text{col1} & \text{row1} \times \text{col2} \\ \text{row2} \times \text{col1} & \text{row2} \times \text{col2} \end{pmatrix}$$

$$= \begin{pmatrix} -1 \times 0 + 0 \times 1 + 1 \times 3 & -1 \times 3 + 0 \times 2 + 1 \times 1 \\ -1 \times 0 + 1 \times 1 + 3 \times 3 & -1 \times 3 + 1 \times 2 + 3 \times 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 10 & 2 \end{pmatrix}$$

Notice that $AB \neq BA$

It also holds that $\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$

Inverse matrix: A inverse matrix of $\boldsymbol{A}$ (hereby $\boldsymbol{A}$ is a square matrix) is denoted by $\boldsymbol{A}^{-1}$ such that $\boldsymbol{AA^{-1}} = \boldsymbol{I} = \boldsymbol{A^{-1}A}$. A matrix can have no inverse matrix, that kind of matrix is called a **singular matrix**.

The inverse of a matrix is an important stuff that we will talk about later. If you do not understand det notation and singular matrix, its okay.

System of equations $A\mathrm{x} = \mathrm{b}$

Consider n linear equations with n variables.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_n \end{cases}$$

we denote that

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ a_{m1} & \vdots & \vdots & a_{mn} \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Here notice that **x** and **b** can be vectors. n-dimesional vectors are $n \times 1$ martices.

the equations can be rewritten into

$$\boldsymbol{A}\mathbf{x} = \mathbf{b}$$

by multiplying $\boldsymbol{A}^{-1}$ to both sides it can be easily discovered that

$$\mathbf{x} = \boldsymbol{A}^{-1}\mathbf{b}$$

Note: this equation hides an important idea, can you use this equation to solve for $\boldsymbol{A}^{-1}$?

Here we can have a good way of judging whether $\boldsymbol{A}$ is singular.

claim. A square matrix is singular if non-zero vector $\mathbf{x}$ ($\mathbf{x} \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}$) such that $\boldsymbol{A}\mathbf{x} = 0$ exists.

e.g: judge whether matrix $\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ is singular

solution. Consider the system of equations

$$\begin{cases} 3x + y = 0 \\ 6x + 2y = 0 \end{cases}$$

we can easily observe that there exists a non-zero solution where $x = -1, y = 3$. therefore

$$\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

According to the claim, the matrix is singular.

Gauss Elimination

Gauss invented a way of solving a system of equations that has n variables.

e.g use gauss elimination to solve

$$\begin{cases} x + y + 2z = 1 \\ 3x + 4y + 6z = 3 \\ -2x + 3y - 3z = -1 \end{cases}$$

solution: first we rewritten the whole equation into form $\boldsymbol{A}\mathbf{x} = \mathbf{b}$

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

Then we write LHS into argumented matrix form

$$\boldsymbol{A_M} = \begin{pmatrix} 1 & 1 & 2 & : & 1 \\ 3 & 4 & 6 & : & 3 \\ -2 & 3 & -3 & : & -1 \end{pmatrix}$$

Also can be written as

$$\boldsymbol{A_M} = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 3 & 4 & 6 & 3 \\ -2 & 3 & -3 & -1 \end{array}\right)$$

(A-level use the former one)

you can see that we just ignore the **x** vector and write the coefficents together, this is because variables have no significance to the elimination process and this form is better-looking.

now I denote each row as R_n, the element of $\boldsymbol{A_M}$ can be represented by a_{mn}. we now set the pivot as a_{11} (at the n-th process, the n-th pivot is usually a_{nn}). The first pivot is usually 1.

Note: you can freely move these rows because you're only solving the equation, not finding the inverse of a matrix. However I personally suggest you don't move rows because this will make you confuse.

Now in order to elimination the first column, we consider make $R_2 \to R_2 - 3R_1$, this gives us.

$$\begin{pmatrix} 1 & 1 & 2 & : & 1 \\ 0 & 1 & 0 & : & 0 \\ -2 & 3 & -3 & : & -1 \end{pmatrix}$$

now a_{22} is the second pivot. we consider make $R_3 \to R_3 + 2R_1$

$$\begin{pmatrix} 1 & 1 & 2 & : & 1 \\ 0 & 1 & 0 & : & 0 \\ 0 & 5 & 1 & : & 1 \end{pmatrix}$$

Now we consider $R_3 \to R_3 - 5R_2$

$$\begin{pmatrix} 1 & 1 & 2 & : & 1 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

Here you can end the process because we have achieved the *echelon form* on the left side.

An *echelon form* is a kind of a matrix such that

$$\begin{pmatrix} \square & * & * & * & \ldots \\ 0 & \square & * & * & \ldots \\ 0 & 0 & \square & * & \ldots \\ 0 & 0 & 0 & \square & \ldots \end{pmatrix}$$

here $\square$ denotes non-zero element, $*$ can be any element.

if echelon form cannot be achieved during the elimination process, then A is singular

now rewrite the system of equations after gauss elimination.

$$\begin{cases} x + y + 2z = 1 \\ -y = 0 \\ z = 1 \end{cases}$$

Now we can easily solve the equation

$$\begin{cases} x = -1 \\ y = 0 \\ z = 1 \end{cases}$$

The determinant $\det(\boldsymbol{A}_{n \times n})$ can be calculated with $\boldsymbol{A}$'s echelon form $\boldsymbol{E} = (e_{ij})$

$$\det(\boldsymbol{A}) = \prod_{k=1}^{n} e_{kk} \times (-1)^{\frac{n(n-1)}{2}}$$

Gauss-Jordan's idea

Gauss-Jordan's idea is useful for finding a matrix's inverse. the core idea is to create an argumented matrix form such that

$$\left(\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{I} \end{array} \right)$$

After gauss's elimination, the result should look like.

$$\left(\begin{array}{c:c} \boldsymbol{I} & \boldsymbol{A}^{-1} \end{array}\right)$$

e.g Find the inverse of matrix :

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 7 & 8 \\ 5 & 1 & 9 \end{pmatrix}$$

solution:

Start with the augmented matrix [A ⋮ I]:

$$\begin{pmatrix} 1 & 2 & 3 & \vdots & 1 & 0 & 0 \\ 4 & 7 & 8 & \vdots & 0 & 1 & 0 \\ 5 & 1 & 9 & \vdots & 0 & 0 & 1 \end{pmatrix}$$

$R_2 \to R_2 - 4R_1, R_3 \to R_3 - 5R_1$

$$\begin{pmatrix} 1 & 2 & 3 & \vdots & 1 & 0 & 0 \\ 0 & -1 & -4 & \vdots & -4 & 1 & 0 \\ 0 & -9 & -6 & \vdots & -5 & 0 & 1 \end{pmatrix}$$

$R_2 \to -R_2$

$$\begin{pmatrix} 1 & 2 & 3 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 4 & \vdots & 4 & -1 & 0 \\ 0 & -9 & -6 & \vdots & -5 & 0 & 1 \end{pmatrix}$$

$R_3 \to R_3 + 9R_2$

$$\begin{pmatrix} 1 & 2 & 3 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 4 & \vdots & 4 & -1 & 0 \\ 0 & 0 & 30 & \vdots & 31 & -9 & 1 \end{pmatrix}$$

$R_3 \to \frac{1}{30}R_3$

$$\begin{pmatrix} 1 & 2 & 3 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 4 & \vdots & 4 & -1 & 0 \\ 0 & 0 & 1 & \vdots & \frac{31}{30} & -\frac{3}{10} & \frac{1}{30} \end{pmatrix}$$

$R_1 \to R_1 - 3R_3, R_2 \to R_2 - 4R_3$

$$\begin{pmatrix} 1 & 2 & 0 & : & -\frac{7}{10} & \frac{9}{10} & -\frac{1}{10} \\ 0 & 1 & 0 & : & -\frac{7}{15} & \frac{1}{5} & -\frac{2}{15} \\ 0 & 0 & 1 & : & \frac{31}{30} & -\frac{3}{10} & \frac{1}{30} \end{pmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\begin{pmatrix} 1 & 0 & 0 & : & -\frac{7}{30} & \frac{1}{2} & \frac{1}{30} \\ 0 & 1 & 0 & : & -\frac{7}{15} & \frac{1}{5} & -\frac{2}{15} \\ 0 & 0 & 1 & : & \frac{31}{30} & -\frac{3}{10} & \frac{1}{30} \end{pmatrix}$$

Therefore:

$$\boldsymbol{A}^{-1} = \begin{pmatrix} -\frac{7}{30} & \frac{1}{2} & \frac{1}{30} \\ -\frac{7}{15} & \frac{1}{5} & -\frac{2}{15} \\ \frac{31}{30} & -\frac{3}{10} & \frac{1}{30} \end{pmatrix}$$

Determinants

determinant of a 2×2 matrix

$$\det(\boldsymbol{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

determinant of a 3×3 matrix

$$\det(\boldsymbol{A}) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + dhc - gec - bdi - hfa$$

$$\det(\boldsymbol{AB}) = \det(\boldsymbol{BA}) = \det(\boldsymbol{A}) \times \det(\boldsymbol{B})$$

Matrix transformations

In general, a two-way stretch of scale factor a in the x-direction and scale factor b in the y-direction is represented by the matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

In general, a shear parallel to the x-axis, with a shear factor of k and parallel to the y-axis, with a shear factor of l, is represented by the matrix $\begin{pmatrix} 1 & k \\ l & 1 \end{pmatrix}$.

In general, a reflection in the line $y = x \tan\theta$ (where θ is the angle the line makes with the x-axis) is represented by the matrix $\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$.

For a transformation matrix A, the value of the determinant, $\det(A)$, is the scale factor of the enlargement of the area from the original shape to the image.

A point is invariant if it does not move under matrix multiplication.

The following transformations are for 2×2 matrices.

Transformation	Matrix
Stretch by a scale factor of factor k in the x-direction	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
Stretch by a scale factor of factor k in the y-direction	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Enlargement with centre of enlargement the origin by a scale factor of factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
Reflection in the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection in the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection in the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Rotation about the origin by θ in the anticlockwise direction	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

The following transformations are for 3×3 matrices.

Transformation	Matrix
Rotation about the x-axis by angle θ in the anticlockwise direction	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
Rotation about the y-axis by angle θ in the anticlockwise direction	$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$
Rotation about the z-axis by angle θ in the anticlockwise direction	$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Enlargement with centre of enlargement the origin by a scale factor of factor k	$\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$

Invariant lines:

For 2-dimensional cases, use $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ mT \end{pmatrix}$ to determine two equations of the form $at + bmt = T, ct + dmt = mT$. Divide to get $\frac{a+bm}{c+dm} = \frac{1}{m}$, then solve for value(s) of m to find the invariant line(s) of the transformation in the form $y = mx$.

(3-dimensional case is out of syallbus)