Rational functions

In this chapter we discuss the properties of rational functions (functions that looks like $f(x) = \frac{p(x)}{q(x)}$ where $p(x), q(x)$ are polynomials) and how to draw its graph.

The *5-step* method to draw a rational function

1. Determine whether this is even or odd function.
2. Find the intercepts.
3. Find the asymptotes
4. Find the stationary points
5. Draw it!

Odd/Even function

An even function is an function such that $f(-x) = f(x)$ so it is symmetry to y-axis

An odd function is an function such that $f(-x) = -f(x)$ so it is symmetry to the origin point.

(These save u $\frac{1}{2}$ effort)

Asymptotes

Horizontal Asymptotes: if

$$\lim_{x \to \pm\infty} f(x) = a \text{ or } b$$

Then $y = a$ and $y = b$ are the horizontal asymptotes of $f(x)$ (Apply L'hopitals)

Vertical asymptotes: if

$$\lim_{x \to c^+} f(x) = \pm\infty \text{ or } \lim_{x \to c^-} f(x) = \pm\infty$$

then $x = c$ is the vertical asymptote of $f(x)$

Often, vertical asymptote of $f(x)$ is the zero of $q(x)$

Oblique Asymptotes: This only happens when $\deg(p(x)) - \deg(q(x)) = 1$

$$\lim_{x \to \pm\infty} [f(x) - mx + c] = 0$$

Then $y = mx + c$ is the oblique asymptote of $f(x)$, most times the quotient of $\frac{p(x)}{q(x)}$ is the oblique asymptote of $f(x)$

Remember: horizontal asymptote and oblique asymptote cannot exist at the same time.

Questions

1. Sketch $\frac{x^2+x-3}{1+x-x^2}$
2. The curve C has equation $y = \frac{x^2+x-1}{x-1}$

   (a) Find the equation of asymptotes of C

   (b) show that there's no point on C for which $1 < y < 5$

Trick: Table function of your calculator (Casio fx-991 CW)

when you have trouble discovering the asymptotes, you can use the "define f(x)" in your calculator and plug-in values to verify.

My calculator (An CAIE Approved Casio fx-991 CW) has a table function which makes it more easy to see the tendency and monotonicity, but it's okay to not have it.

Variations of functions

1. $\boldsymbol{y = f(|x|)}$ **and** $\boldsymbol{y = |f(x)|}$: $y = |f(x)|$ is $f(x)$ but all place where $y < 0$ is symmetrical to x-axis, $y = f(|x|)$ is $f(x)$ but the part where $x < 0$ is replaced with the part $x > 0$ symmetrical to y-axis
2. $\boldsymbol{y = \frac{1}{f(x)}}$: When sketch $y = \frac{1}{f(x)}$, you need to remember that
   - Local maximum becomes local minimum
   - zero point becomes vertical asymptote
   - Monotonicity changes inversely
   - $$\begin{cases} f(x) > 1 \Rightarrow \frac{1}{f(x)} \in (0,1) \\ f(x) \in (0,1) \Rightarrow \frac{1}{f(x)} > 1 \\ f(x) < -1 \Rightarrow \frac{1}{f(x)} \in (-1,0) \\ f(x) \in (-1,0) \Rightarrow \frac{1}{f(x)} < -1 \end{cases}$$

3. $\boldsymbol{y^2 = f(x)}$

Solving the equation you get $y = \pm\sqrt{f(x)}$, so you can actually just sketch $y = \sqrt{f(x)}$ and then do the reflection of x-axis.

Some important stuff to remember:

1. $y^2 = f(x)$ intersects with $y = f(x)$ in $y = 0$ and $y = 1$
2. The stationary point on $y = f(x)$ with coordinate (a, b) will have a coordinate of $\left(a, \pm\sqrt{b}\right)$ in $y^2 = f(x)$
3. if $f(x) = 0$ and $f'(x) \neq 0$ then the tangent of the curve in this interval is parallel to y-axis
4. $$\begin{cases} 0 < f(x) < 1 \Rightarrow \sqrt{f(x)} > f(x) \\ f(x) > 1 \Rightarrow \sqrt{f(x)} < f(x) \end{cases}$$